

# Hamilton-Jacobi formalism of the massive Yang-Mills theory revisited

Dumitru Baleanu<sup>1</sup>,

Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University-06530, Ankara , Turkey

## Abstract

Using Hamilton-Jacobi formalism we investigated the massive Yang-Mills theory on both extended and reduced phase-space. The integrability conditions were discussed and the actions were calculated.

## 1 Introduction

Since Dirac initiated [1] the Hamiltonian formulation of constrained systems, the second class systems were subjected to an intense debate [2]. Based on Caratheodory's idea [3] Hamilton-Jacobi formalism (HJ) for constrained systems was initiated [4, 5] and developed in [6, 7, 8, 9]. Specific for this formalism is the fact that we started with many "Hamiltonians" and imposing the integrability conditions we obtain the action. It was proved that (HJ) stabilization algorithm gives the same results as Dirac's procedure although there are two different ways of obtaining the constraints. For second-class constrained systems (HJ) formalism has many subtle aspects which should be clarified. First of all the system corresponding to the total differential equations are not integrable. To make it integrable we must modify it. The first option is to solve the total differential equations by adding the new set of equations obtained in the process of variation. This procedure is working properly if the form of the equations is not so complicated and it is extremely difficult to apply it if the equations are non-linear ones. For these reasons new ways of making the system integrable should be developed without solving the total differential equations. One way is to transform the system, by reducing or extending the phase-space, in such a manner that the modified "Hamiltonians" have a physical interpretation from (HJ) point of view [10]. In this paper we will present two methods of making an integrable system from (HJ) point of view. The first way is based on the chain method and the second one is deeply related to the powerful formalism Batalin-Fradkin-Vilkovisky (BFT) [11] or its modified versions [12], [13]. To illustrate the methods we investigated the gauge non invariant massive Yang-Mills theory, which is one of the important models admitting non-linear second-class constraints.

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<sup>1</sup>On leave of absence from Institute of Space Sciences, P.O BOX, MG-23, R 76900 Magurele-Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr

The plan of the paper is as follows:

In Sec. 2 (HJ) formalism and the chain method were briefly presented. In Sec.3 the massive Yang-Mills theory is analyzed within (HJ) formalism. Our conclusions are given in Sec. 4.

## 2 Hamilton-Jacobi formalism

Let us assume that the Lagrangian  $L$  is singular and the Hessian supermatrix has rank  $n-r$ . The Hamiltonians to start with are

$$H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha, \quad (1)$$

where  $\alpha, \beta = n-r+1, \dots, n, a = 1, \dots, n-r$ . The usual Hamiltonian  $H_0$  is defined as

$$H_0 = -L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a) + p_a w_a + \dot{q}_\mu p_\mu \big|_{p_\nu = -H_\nu, \nu = 0, n-r+1, \dots, n}. \quad (2)$$

which is independent of  $\dot{q}_\mu$ . Here  $\dot{q}_a = \frac{dq_a}{d\tau}$ , where  $\tau$  is a parameter. The equations of motion are obtained as total differential equations in many variables as follows

$$\begin{aligned} dq_a &= (-1)^{P_a + P_\alpha P_\alpha} \frac{\partial_r H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -(-1)^{P_a P_\alpha} \frac{\partial_r H'_\alpha}{\partial q_a} dt_\alpha, \\ dp_\mu &= -(-1)^{P_\mu P_\alpha} \frac{\partial_r H'_\alpha}{\partial t_\mu} dt_\alpha, \quad \mu = 1, \dots, r, \end{aligned} \quad (3)$$

$$dz = (-H_\alpha + (-1)^{P_a + P_\alpha P_\alpha} p_a \frac{\partial_r H'_\alpha}{\partial p_a}) dt_\alpha, \quad (4)$$

where  $z = S(t_\alpha, q_a)$  and  $P_i$  represents the parity of  $a_i$ .

On the surface of constraints the system of differential equations(3) is integrable if and only if

$$[H'_\alpha, H'_\beta] = 0. \quad (5)$$

As we can say from (5) if the system is second class in Dirac's classification, then it is not integrable. For second class constrained systems some of the "Hamiltonians" are not in the form (1) and some of them are not in involution. In the following we will solve these problems using two different techniques working in reduced phase-space and extended phase-space respectively.

## 2.1 The chain method

Let us assume that the local Lagrangian density  $L$  is singular and admits only one second-class constraints [14]  $\phi_1(\vec{x}), \phi_2(\vec{x}), \dots, \phi_{2n}(\vec{x})$  and only  $\phi_1(\vec{x})$  is primary. In this method we have infinite number of constraint equations. If we calculate the bracket between constraints we will construct the matrix  $M_{ij} = \{\phi_i, \phi_j\}$  as

$$M_{ij}(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -\alpha \\ 0 & 0 & 0 & \cdots & 0 & \alpha & * \\ 0 & 0 & 0 & & -\alpha & * & * \\ \vdots & & & & & & \vdots \\ 0 & 0 & \alpha & & 0 & * & * \\ 0 & -\alpha & * & & * & 0 & * \\ \alpha & * & * & \cdots & * & * & 0 \end{pmatrix} \delta(\vec{x} - \vec{y}), \quad (6)$$

where  $\{\phi_1, \phi_{2n}\} = -\alpha$ .

To simplify the problem and to explain the method we consider the two-chain case. Let us assume that the set of the constraints in involution is given  $\{\chi_1, \chi_2, \dots, \chi_r; \psi_1, \psi_2, \dots, \psi_q\}$ . To eliminate the other constraints we transform the Hamiltonian as [14]

$$H'' = H_c + \frac{1}{2} \chi^T \alpha^{-1} \chi, \quad (7)$$

where

$$\chi = \begin{pmatrix} \chi_{r+1} \\ \psi_{q+1} \end{pmatrix}. \quad (8)$$

Let us assume that for field theory only a primary constraint  $\Phi_1$  generates  $2n-1$  constraints denoted by  $\Phi_\alpha, \alpha = 2, \dots, 2n$ . If the corresponding extended Hamiltonian has the following form

$$H''' = H_c + \frac{1}{2} \int d\vec{x} \alpha^{-1}(\vec{x}) \Phi_{n+1}^2(\vec{x}), \quad (9)$$

then half of the constraints are eliminated and the resulting system is a first-class one (for more details see Ref.[14]).

If, for a given second-class constrained system we will find a subset of constraints in involution and in the form (1), then due to (5) the system is integrable from (HJ) point of view and we can calculate the action.

## 3 Massive Yang-Mills theory

The Lagrangian density is given by

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} m^2 A_\mu^a A^{a\mu}, \quad (10)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \quad (11)$$

In addition we use the following notations and conventions

$$[T^a, T^b] = if^{abc}T^c, \text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, (T^a)^{bc} = if^{abc}. \quad (12)$$

The canonical Hamiltonian density corresponding to (10) is

$$H_c = \frac{1}{2}(\pi_i^a)^2 + \pi^{ai}\partial_i A_0^a - gf^{abc}\pi^{ai}A_0^b A_i^c + \frac{1}{4}F_{ij}^a F^{aij} - \frac{1}{2}m^2 A_i^a A^{ai} - \frac{1}{2}m^2 A_0^a A^{a0}. \quad (13)$$

Let us apply the chain method to investigate the constraints of the system. The primary constraint is

$$\phi_1^a = \pi_0^a = 0 \quad (14)$$

and the total Hamiltonian is given by

$$H_T = \int d^3x (H_c + \lambda^a \pi_0^a), \quad (15)$$

where  $\lambda^a$  are Lagrange multipliers. The consistency condition of  $\phi_1^a$  gives another constraint

$$\phi_2^a = (D_i \pi^i)^a + m^2 A_0^a, \quad (16)$$

where  $(D_i \pi^i)^a = \partial_i \pi^{ai} + gf^{abc}A_i^b \pi^{ci}$ . If we impose the consistency condition for  $\phi_1^a$  we obtain an equation for  $\lambda^a$ . The system is second-class and the Poisson algebra of  $\phi_0^a$  and  $\phi_1^a$  is given by

$$\begin{aligned} \{\phi_0^a(x), \phi_0^b(y)\} &= 0, \{\phi_0^a(x), \phi_1^b(y)\} = -m^2 \delta^{ab} \delta(\vec{x} - \vec{y}), \\ \{\phi_1^a(x), \phi_1^b(y)\} &= gf^{abc}(D_i \pi^i)^c \delta(\vec{x} - \vec{y}). \end{aligned} \quad (17)$$

In (HJ) formalism the "Hamiltonian" densities to start with are

$$H'_0 = p_0 + H_c, H_1'^a = \pi_0^a. \quad (18)$$

If we consider the variations of  $H_1'^a$  we obtain another "Hamiltonian"

$$H_2^a = (D_i \pi^i)^a + m^2 A_0^a. \quad (19)$$

The variation of (19) gives an equation for  $\dot{A}_0^a$  which is nothing than the form of Lagrange multipliers  $\lambda^a$  from the previous method. From (18) and 19 we deduce that the "Hamiltonians" are not in the form required by (5). At this stage we will transform the "Hamiltonians" such that they will be in involution on the reduced phase-space. We choose  $H_1'^a = \pi_0^a$  and we transform  $H'_0 = p_0 + H_c$  as follows

$$\tilde{H}'_0 = p_0 + H_c - \frac{1}{2} \frac{1}{m^2} \int \left( d\vec{x} \left[ \sum_{a=1}^{a=n} (D_i \pi^i)^a + m^2 A_0^a \right]^2 \right). \quad (20)$$

In this manner we obtain an integrable system corresponding to  $\tilde{H}'_0$  and  $H_1'^a$ . The corresponding action of these two "Hamiltonians" is given by

$$\begin{aligned} S = \int \{ & \frac{-1}{4} F_{ij}^a F^{aij} + \frac{1}{2} \pi^{ai} \pi_i^a + \frac{1}{2} m^2 A_0^a A^{a0} - \frac{1}{2} m^2 A_i^a A^{ai} \\ & - \frac{1}{2m^2} \sum_{a=1}^{a=n} [(D_i \pi^i)^a + m^2 A_0^a][(D_i \pi^i)^a - m^2 A_0^a] \} d\vec{x}. \end{aligned} \quad (21)$$

### 3.1 Modified BFT conversion

Another method to make the "Hamiltonians" in involution and to keep their physical significance from (HJ) point of view is based on the use of (BFT) formalism [11]. Let us suppose that for a system with  $N$  degrees of freedom  $T_a$  second-class constraints exists, where  $a = 1, \dots, M < 2N$  and

$$\{T_a, T_b\} = \Delta_{ab} \quad (22)$$

with  $\det(\Delta_{ab}) \neq 0$ . The aim of the formalism is to transform the above constraints into first-class ones by adding auxiliary variables  $\eta^a$  [11]. In addition the auxiliary variables fulfills a symplectic algebra

$$\{\eta^a, \eta^b\} = \omega^{ab}, \quad (23)$$

where  $\omega^{ab}$  represents a constant quantity subjected to the restriction  $\det(\omega^{ab}) \neq 0$ . For (HJ) formalism the form of the first-class constraints  $\hat{T}_a = \hat{T}_a(q, p, \eta)$  is crucial. In [11] the form of the constraints in involution was required to be

$$\hat{T}_a = \sum_{n=0}^{\infty} T_a^{(n)}, \quad (24)$$

where  $T_a^{(n)}$  represents the  $n^{th}$  order in  $q$  and  $\eta$ . To keep the physical significance of the "Hamiltonians" we require them to be in the form

$$\hat{H}'_{\alpha} = \hat{p}_{\alpha} + \hat{H}_{\alpha} \quad (25)$$

on the extended phase-space. In (25) the involved quantities are functions on the extended phase-space. The key point is to calculate the first correction  $T_a^{(1)} = X_{ab}(q, p)\eta^b$  in such a way to obtain  $\hat{T}_a$  in the form given by (25). Taking into account that for the massive Yang-Mills theory we have

$$\omega_{A,B}^{ab}(x, y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^{ab} \delta(x - y). \quad (26)$$

after calculations we found that

$$X_{AB}^{ab}(x, y) = \begin{pmatrix} m^2 \delta^{ab} & 0 \\ -\frac{1}{2} g f^{abd} (D_i \pi^i)^d & \delta^{ab} \end{pmatrix}. \quad (27)$$

Following (BFT) formalism we found the constraints in involution as

$$\hat{\phi}_1^a = \pi_0^a + m^2 \eta^{1a}, \hat{\phi}_2^a = \eta^{2a} + (D_i \pi^i)^a + m^2 A_0^a - \sum_{n=1}^{\infty} [(g \bar{\eta})^n]^{ab} (D_i \pi^i)^b, \quad (28)$$

where

$$\bar{\eta}^{ac} = f^{abc} \eta^{1b}. \quad (29)$$

Using the technique described in [12, 13] we will construct the involutive forms of the initial fields. After a tedious calculations we obtained

$$\begin{aligned}\hat{\pi}_i^a &= \pi_i^a + \sum_{n=1}^{\infty} (-1)^n \frac{g^n}{n!} (\bar{\eta}^n)^{ac} \pi_i^c, \\ \hat{A}_i^a &= A_i^a + \frac{1}{(n+1)!} \sum_{n=1}^{\infty} (g\bar{\eta}^n)^{ac} D_i \eta^{2c} \\ \hat{A}_0^a &= A_0^a + \frac{1}{m^2} \sum_{n=1}^{\infty} [\eta^{2a} \delta_{1n} + b_n ((g\bar{\eta}^n)^{ac}) (D_i \pi^i)^c],\end{aligned}\tag{30}$$

where

$$b_n = \frac{1}{n} [-b_{n-1} + \frac{1}{(n+1)!}], n > 1, b_1 = \frac{1}{2}.\tag{31}$$

We mention that the our results are different from those obtained in [12, 13]. The involutive Hamiltonian  $\hat{H}_c(\hat{A}_0, \hat{A}_i, \hat{\pi}_i^a)$  is obtained from  $H_c(A_0, A_i, \pi_i^a)$  by making the following substitutions  $A_0 \rightarrow \hat{A}_0, A_i \rightarrow \hat{A}_i, \pi_i^a \rightarrow \hat{\pi}_i^a$ . "Hamiltonians"  $H_1'^a = \hat{\phi}_1^a, H_2'^a = \hat{\phi}_2^a$  and  $H_0'' = p_0 + \hat{H}_c$  are in involution and all of them have physical significance from (HJ) point of view [10]. The corresponding action of  $H_1'^a, H_2'^a$  and  $H_0''$  is given by

$$\begin{aligned}dS &= -m^2 \eta^{1a} d\pi^{0a} - (m^2 A_0^a) d\eta^{1a} + \left\{ \frac{1}{2} (\pi_i^a + \sum_{n=1}^{\infty} (-1)^n \frac{g^n}{n!} (\bar{\eta}^n)^{ac} \pi_i^c)^2 \right. \\ &\quad \left. (\pi_i^a + \sum_{n=1}^{\infty} (-1)^n \frac{g^n}{n!} (\bar{\eta}^n)^{ac} \pi_i^c) \left( \partial_i [b_n (g\bar{\eta}^n)^{ac}] g f^{cdb} A_j^d \pi^{jd} - \partial_j^2 [b_n (g\bar{\eta}^n)^{ac}] \pi^{jc} \right) \right. \\ &\quad \left. - \frac{1}{4} F_{ij}^a F^{aij} + \frac{1}{2} m^2 A_i^a A^{ai} + g f^{abc} \pi^{ai} A_0^b A_i^c + \right. \\ &\quad \left. (A_0^a + \frac{1}{m^2} \sum_{n=1}^{\infty} [\eta^{2a} \delta_{1n} + b_n ((g\bar{\eta}^n)^{ac}) (D_i \pi^i)^c]) \left[ \frac{1}{2} A_0^a - \frac{1}{2m^2} (\sum_{n=1}^{\infty} (-\eta^{2a} \delta_{1n} + b_n ((g\bar{\eta}^n)^{ac}))) \right] \right\} dt.\end{aligned}\tag{32}$$

## 4 Conclusions

Second-class constrained systems are problematic for (HJ) formalism because the corresponding system of total differential equations are not integrable. In this paper we presented another way to make the "Hamiltonians" in involution, and implicitly to obtain an integrable system, without solving the equations of motion. The basic idea was to keep the physical significance of the modified "Hamiltonians". In the first part we applied the chain method to make the system corresponding to the massive Yang-Mills theory integrable on the reduced phase-space. Due to the form of the modified constraints all "Hamiltonians" have physical significance from (HJ) point of view. In the second part we used a method based on modified (BFT) formalism and we obtained an integrable system on the extended phase-space. In (BFT) formalism the process of making the constraints in involution is not unique. Using this important property of this approach we followed that way which gives the physical significance for all "Hamiltonians" on the extended phase-space. The obtained results are different from those presented in [12, 13]. In both cases the action corresponding to the involutive "Hamiltonians" was calculated.

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